The Analysis on the Running Time of the Generalized Quantum Search Hamiltonian : $O(N^{1/4})$ and $O(1)$ Time Resources

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Abstract

Farhi et al. suggested the analogue quantum search Hamiltonian and Fenner also proposed the intuitive quantum search Hamiltonian. Recently the generalized quantum search Hamiltonian containning hamiltonians of Farhi et al. and Fenner was presented in quant-ph/0110020. In this letter, we analyze the running time of the generalized quantum search Hamiltonian. Our analysis displays two surprising results. The first is the exponential speedup($T = O(N^{1/4})$), and the next $O(1)$ time resource.

In 1996, Farhi et al. suggested the analogue quantum search algorithm based on Hamiltonian evolution.[1] The Hamiltonian to solve the search problem is $H_{af} = Ed(|w\rangle\langle w| + |\psi\rangle\langle\psi|)$. In the Hamiltonian H_{af}, $|w\rangle$ is the target state that we have to find, $|\psi\rangle$ is the initial state that is a superposition of N states, E is a constant in unit of energy , and d is a constant. Fenner [2] also proposed the quantum search Hamiltonian $H_{if} = -2iEx(|w\rangle\langle\psi| - |\psi\rangle\langle w|)$, where $x = \langle w|\psi\rangle \approx \frac{1}{\sqrt{N}}$. The Hamiltonian H_{if} is called the intuitive quantum search Hamiltonian.

Recently, the generalized quantum search Hamiltonian including hamiltonians of Farhi et al. and Fenner was presented.[3] The Hamiltonian is as follows:

$$H_{yj} = E[d(|w\rangle\langle w| + |\psi\rangle\langle\psi|) + r(e^{i\phi}|w\rangle\langle\psi| + e^{-i\phi}|\psi\rangle\langle w|)] \tag{1}$$

where, r is a constant, and ϕ is a phase. The initial state can be written as $|\psi\rangle = x|w\rangle + \sqrt{1-x^2}|\beta\rangle$, where $|\beta\rangle$ is the orthogonal complement of the state $|w\rangle$. If the phase is given by $\phi = n\pi$(n is an integer), then the Hamiltonian finds the target state with probability one, but if not, the probability to find the target becomes $1 - O(x^2) \approx 1 - O(1/N)$.

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In this letter, we analyze the running time T of the generalized quantum search Hamiltonian H_{yj}. The running time is exactly given by

$$T_{yj} = \frac{\pi}{2E} \frac{1}{\sqrt{(d^2 - r^2 sin^2\phi)x^2 + 2drcos\phi x + r^2}} \tag{2}$$

We will consider the running time T_{yj} in every possible cases. Before doing it, we note that the running time of the analogue quantum search Hamiltonian H_{af} is

$$T_{af} = \frac{\pi}{2E} \frac{1}{dx} \approx \frac{\pi}{2Ed} \sqrt{N} \tag{3}$$

The running time T_{af} can be given as a particular case of T_{yj}, by letting $r = 0$.

$Case 1.$ $|d| > |r|$

In this case, we have $(d^2 - r^2 sin^2\phi)x^2 + 2drcos\phi x + r^2 > dx$ if $\phi \in [-\pi/2, \pi/2]$. Therefore, the running time is, for $\phi \in [-\pi/2, \pi/2]$

$$T_{yj} = \frac{\pi}{2E} \frac{1}{\sqrt{(d^2 - r^2 sin^2\phi)x^2 + 2drcos\phi x + r^2}} < \frac{\pi}{2E} \frac{1}{dx} = T_{af}$$

Thus, the square-root speedup is presented. And we can see that the generalized quantum search Hamiltonian finds the target state faster than the analogue quantum search Hamiltonian does.

$Case 2.$ $|d| < |r|$

In this case, we have by some manipulation of (2)

$$T_{yj} = \frac{\pi}{2E} \frac{1}{\sqrt{r^2 x^2 (cos\phi + \frac{d}{rx})^2 + (r^2 - d^2)(1 - x^2)}} = O(x^{-1}) \approx O(\sqrt{N}) \tag{4}$$

The square-root speedup is shown. However, we consider the special values of the phase ϕ, in order to find out the minimum value of the running time. First, if $\phi = cos^{-1} \pm \frac{\sqrt{r^2 - d^2}}{r}$ is chosen, then the time becomes

$$T_{yj} = \frac{\pi}{2E} \frac{1}{\sqrt{2dx\sqrt{r^2 - d^2} + r^2}} = O(x^{-1/2}) \approx O(N^{1/4})$$

Surprisingly, the speedup of N to the power of one fourth(exponential speedup) is shown. Next, if $\phi = cos^{-1} \pm \frac{d}{rx}$, which holds only when d is so small that $d \leq rx$, then we have

$$T_{yj} = \frac{\pi}{2E} \frac{1}{\sqrt{(r^2 - d^2)(1 - x^2)}}$$

The running time is getting smaller as the number of states N becomes larger, since $x^2 \approx \frac{1}{N}$. For a sufficiently large N, we have $O(1)$ time resource.

$Case3$. $|d| = |r|$

In this case, the running time is

$$T_{yj} = \frac{\pi}{2E}\frac{1}{r(xcos\phi + 1)} = O(x^{-1}) = O(\sqrt{N})$$

The square-root speedup is also shown. We consider the special case of $\phi = \pm\frac{\pi}{2}$. Then, we have the running time

$$T_{yj} = \frac{\pi}{2E}\frac{1}{r} = constant$$

The wonderful result appears again. That is, the running time is independent of the number of states N. This implies that the constant time resource is enough to perform the quantum search algorithm based on Hamiltonian evolution.
We now classify quantum search Hamiltonians according to the running time of each Hamiltonian.

Type1. The Square-Root Speedup Quantum Search Hamiltonian

$H_1 = E(|w\rangle\langle w| + |\psi\rangle\langle\psi|) + \epsilon(e^{i\phi}|w\rangle\langle\psi| + e^{-i\phi}|\psi\rangle\langle w|)$ with $E > \epsilon$

Type2. The Exponential Speedup Quantum Search Hamiltonian

$H_2 = E(|w\rangle\langle w| + |\psi\rangle\psi\langle|) + \epsilon(e^{i\phi}|w\rangle\langle\psi| + e^{-i\phi}|\psi\rangle\langle w|)$ with $E < \epsilon$ and $\phi = sin^{-1} \pm \frac{E}{\epsilon}$

Type3. The Constant Time Resource Quantum Search Hamiltonian

$H_3 = E[|w\rangle\langle w| + |\psi\rangle\langle\psi| \pm i|w\rangle\langle\psi| \mp i|\psi\rangle\langle w|]$

Type4. The Hamiltonian in the Middle of the Type2 and the Type3

$H_4 = E(|w\rangle\langle w| + |\psi\rangle\psi\langle|) + \epsilon(e^{i\phi}|w\rangle\langle\psi| + e^{-i\phi}|\psi\rangle\langle w|)$ with $E < \epsilon$ and $\phi = cos^{-1}(-E/(\epsilon| < w|\psi > |))$, under the assumption of $E < \epsilon| < w|\psi > |$

In addition, we consider the quantum search Hamiltonian $H_{new} = E(e^{i\phi}|w\rangle\langle\psi| + e^{-i\phi}|\psi\rangle\langle w|)$.[3] This Hamiltonian H_{new} is included in type 1 when the phase ϕ is arbitrarily chosen. However, if the phase is chosen as $\phi = n\pi$, then the running time becomes $T_{new} = \frac{\pi}{2E}\frac{1}{r}$(type3), and moreover, the probability to obtain the target state after the running time is one since the perfect searching condition $\phi = n\pi$ is given. Therefore, we propose that the best quantum search Hamiltonian with the constant running time independent to the number of states N ,performing the perfect search, is

$$H = \pm E(|w\rangle\langle\psi| + |\psi\rangle\langle w|)$$

We have shown that the generalized quantum search Hamiltonian might find the target state with the exponential speedup or in the constant time independent of the states. According to the degree of the speedups of the quantum search Hamiltonians, we have classified the quantum search Hamiltonians into four types. As an example, the analogue and intuitive quantum search Hamiltonians are included in the type1. From type1 to type4, the probability to find the target state after each running time may be $1-O(x^{-2}) \approx 1-O(1/N)$, although their running times are different. To obtain the target state with probability one, the perfect searching condition $\phi = n\pi$ is necessary. The perfect searching is possible for the Hamiltonians of type1, type2, and type4.

The exponential speedup and the $O(1)$ running time are due to the phase ϕ. The probability to obtain the target state after the running time is robust for various values of the phase, but the running time does not. The running time is strongly dependent on the phase. It implies that the quantum search algorithm based on Hamiltonian evolution makes use of the quantum mechanical effects such as superposition principle and the phase interference phenomenon.

Furthermore, it should be noted that our results may be the good example for the quantum complexity theory. The quantum search algorithm based on the generalized quantum search Hamiltonian may provide some novel approach or some intuition to the quantum complexity theory.

References

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[3] J. Bae and Y. Kwon, “Generalized Quantum Search Hamiltonian”, http://xxx.lanl.gov/abs/quant-ph/0110020.